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IMPROVED TIME INTEGRATION FOR PSEUDODYNAMIC TESTS

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SUMMARY

Converting the second-order differential equation to a first-order equation by integrating it with respect to time once as the governing equation of motion for a structural system can be very promising in the pseudodynamic testing. This was originally found and developed by Chang.[1–3] The application of this time-integration technique to the Newmark explicit method is implimented and investigated in this paper. The main advantages of using the integral form of Newmark explicit method instead of the commonly used Newmark explicit method in a pseudodynamic test are: a less-error propagation effect, a better capability in capturing the rapid changes of dynamic loading and in eliminating the adverse linearization errors. All these improvements have been verified by theoretical studies and experimental tests. Consequently, for a same time step this time-integration technique may result in less-error propagation and achieve more accurate test results than applying the original form of Newmark explicit method in a pseudodynamic test due to these significant improvements. Thus, the incorporation of this proposed time-integration technique into the direct integration method for pseudodynamic testings is strongly recommended. © 1998 John Wiley & Sons, Ltd.

KEY WORDS: time integration; linearization error; pseudodynamic test

INTRODUCTION

In the pseudodynamic testing, the dynamic responses of a specimen advance in a step-by-step manner through direct integration. Hundreds or even thousands of numerical time steps may be required in a pseudodynamic test. In this test method, it is unavoidable to use the data obtained from the previous steps to yield the responses for the current time step. Therefore, the errors introduced in each time step are carried over to the subsequent time steps. Furthermore, these errors tend to propagate and accumulate through this feedback procedure from the initiation of the errors to the end of the test.[4–6] Consequently, the test results may significantly deviate from the actual behaviour even though the errors introduced in each time step are relatively small.

The constraint on the size of time step is undesirable since it increases the number of the total time steps and thus the duration of a test. However, the size of time step cannot be chosen arbitrarily. In fact, the size of time step for a pseudodynamic test is limited by

(1) the presence of high-frequency modes,
(2) the rapid variation of resistance and
(3) the rapid changes of dynamic loading.

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The presence of high-frequency modes may lead to a severe stability problem for a conditionally stable explicit method where the product of the highest frequency and the size of time step should be smaller than a certain value.[7] Numerical solution becomes unstable if the product exceeds the limit. This difficulty has been removed after the successful implementations of several unconditionally stable implicit schemes for pseudodynamic testings.[8–12]

In a pseudodynamic test, it is necessary to employ a direct integration method in which the behaviour of the non-linear system is approximated by a series of successively changing linear systems. Hence, large errors may be introduced if the rapid variation of resistance due to material and/or geometric non-linearities is not precisely captured. Especially, an abrupt change of material properties during a time step is not accurately found. Abrupt changes of material properties may occur as the structural member yields or is unloaded. These errors introduced by the assumption that the structural properties remain constant during a time step will be termed 'linearization errors'. Obviously, these linearization errors may lead to a significant drift effect if a large time step is used.

Accurately simulating the external force in a pseudodynamic test is a key factor to achieve reliable test results. The external force is, often, expressed in terms of ground acceleration, which is usually a very jagged function of time. Thus, the size of time step used must be so small that the rapid changes of dynamic loading can be captured. For a low-frequency input, the size of time step might not be limited by the rapid changes of dynamic loading, while for a high-frequency input, a time step much smaller than that required by accuracy consideration might be needed.

In pseudodynamic tests or non-linear dynamic analysis, a small time step has often been adopted to overcome the limitations (2) and (3) generally. Recently, it has been found to be very promising to integrate the equations of motion to overcome these two difficulties[1–3, 13–15] without using a small time step. The procedure involving time integration of equations of motion leads to the integration of external forces and restoring forces. Thus, the jagged character of the dynamic loading can be effectively smoothed out by the integration process. Meanwhile, the time integration of restoring forces also eliminates the adverse linearization errors. This is because that the restoring force developed by the test structure can be almost continuously measured during each time step in a pseudodynamic test. The superior property in eliminating linearization errors and the better capability in capturing the rapid changes of external forces may significantly improve the accuracy of the pseudodynamic test results. The detailed implementation of the proposed time-integration technique on the Newmark explicit method for pseudodynamic testings are presented herein.

NEWMARK EXPLICIT METHOD

In structural dynamics, the equation of motion for a single-degree-of-freedom system can be written as

$$m\ddot{u} + c\dot{u} + ku = f \tag{1}$$

where m, c, and k are the mass, viscous damping and stiffness, respectively, u, $\dot{u}$, and $\ddot{u}$ are the displacement, velocity and acceleration, respectively, and f is the external force. For a non-linear system, it is more appropriate to replace the product of stiffness and displacement ku by the restoring force r since the stiffness is no longer a constant in the non-linear range. If the equation of motion is integrated with respect to time once from 0 to t, it becomes

$$m\dot{u} + cu + k\bar{u} = \bar{f} \tag{2}$$

if $m\dot{u}(0) + cu(0) + k\bar{u}(0) = \bar{f}(0)$ is assumed to be satisfied. In equation (2), $\bar{u}$ represents the time integral of displacement and $\bar{f}$ is the time integral of external force. In consistence with the use of r in representing ku, the symbol $\bar{r}$ is introduced to denote the time integral of restoring force for a non-linear system. Thus, for any given initial conditions $u(0)$ and $\dot{u}(0)$, the time integral of restoring force $\bar{r} = k\bar{u}(0)$ can be evaluated from the integrated equation of motion directly. The value of $\bar{f}(0)$ for an earthquake input can be taken to be zero since

in the first few time steps the ground acceleration are almost zero. On the other hand, if an external force can be expressed as an analytical function, for example $f(t) = A\sin(\omega t)$, then, the analytical form of the integrated external force can be expressed as $\bar{f}(t) = -(A/\omega)\cos(\omega t)$ and hence the value of $\bar{f}(0)$ is found to be $-(A/\omega)$ at $t = 0$.

In order to illustrate the application of the proposed time-integration technique in the pseudodynamic testing, it is necessary to select a direct integration method. For the purposes of discussing the basic techniques and facilitating a straightforward implementation for pseudodynamic testings, the Newmark explicit method[16] is considered in this study. For a single-degree-of-freedom system, the formulation of this algorithm can be described as follows:

$$\begin{aligned} m\Delta a_{n+1} + c\Delta v_{n+1} + \Delta r_{n+1} &= \Delta f_{n+1} \\ \Delta d_{n+1} &= (\Delta t)v_n + \tfrac{1}{2}(\Delta t)^2 a_n \\ \Delta v_{n+1} &= \tfrac{1}{2}(\Delta t)(a_n + a_{n+1}) \end{aligned} \tag{3}$$

where

$$\begin{aligned} \Delta a_{n+1} &= a_{n+1} - a_n \\ \Delta v_{n+1} &= v_{n+1} - v_n \\ \Delta d_{n+1} &= d_{n+1} - d_n \\ \Delta r_{n+1} &= r_{n+1} - r_n \\ \Delta f_{n+1} &= f_{n+1} - f_n \end{aligned} \tag{4}$$

The terms a_n, v_n, d_n, r_n and f_n are the approximate solutions of acceleration, velocity, displacement, restoring force and external force, respectively. The subscript (n) indicates the time step at $t = n(\Delta t)$. The Newmark explicit method is a conditionally stable explicit method whose stability limit is 2.[7]

In a pseudodynamic test, m and c are analytically prescribed and the restoring force r is measured experimentally at the end of each time step, whereas a, v, and d are computed at each time step.[17] The following steps are a summary of the Newmark explicit algorithm starting at the beginning of step $n + 1$:

(1) Compute the displacement increment for the next step Δd_{n+1}, using the second equation in equation (3).
(2) Impose the computed displacement increment on the specimen.
(3) Measure the restoring force r_{n+1} at the end of $(n + 1)$th step.
(4) Compute the acceleration a_{n+1} and the velocity v_{n+1}.
(5) Repeat.

INTEGRAL FORM OF NEWMARK EXPLICIT METHOD

The Newmark explicit method is also applicable to solve the integrated equation of motion as shown in equation (2) with some modifications.[2,3] It can be reformulated as

$$\begin{aligned} m\Delta v_{n+1} + c\Delta d_{n+1} + \Delta \bar{r}_{n+1} &= \Delta \bar{f}_{n+1} \\ \Delta s_{n+1} &= (\Delta t)d_n + \tfrac{1}{2}(\Delta t)^2 v_n \\ \Delta d_{n+1} &= \tfrac{1}{2}(\Delta t)(v_n + v_{n+1}) \end{aligned} \tag{5}$$

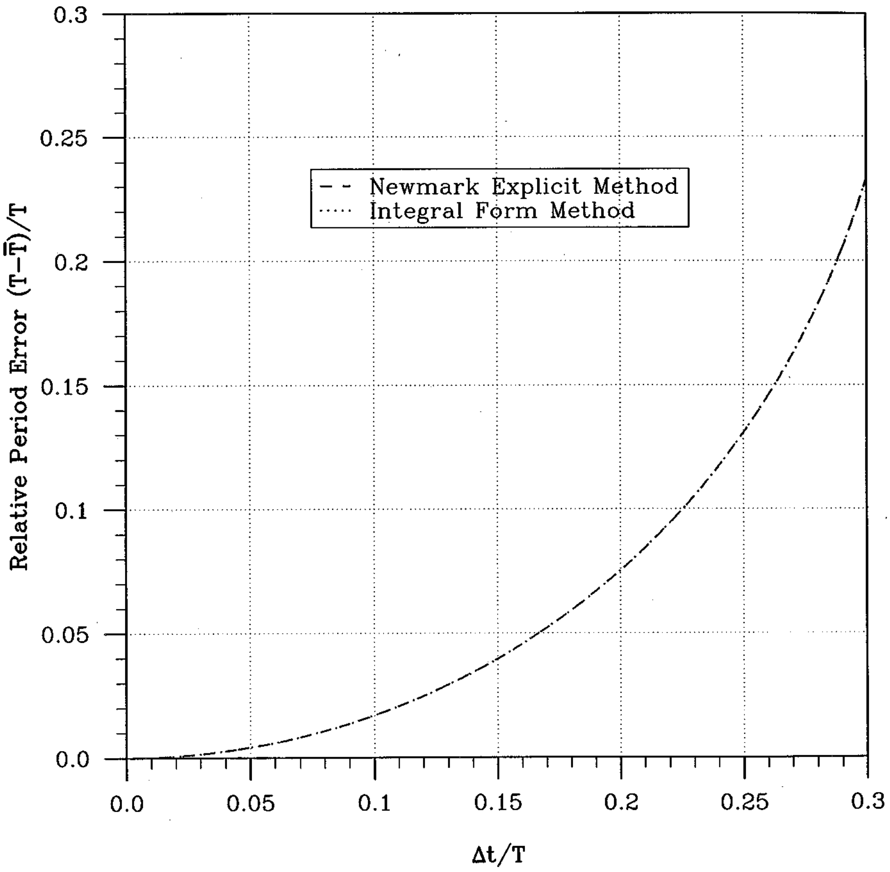

Figure 1. Relative period error versus $\Delta t/T$

where

$$\Delta s_{n+1} = s_{n+1} - s_n, \quad \Delta \bar{r}_{n+1} = \bar{r}_{n+1} - \bar{r}_n, \quad \Delta \bar{f}_{n+1} = \bar{f}_{n+1} - \bar{f}_n \tag{6}$$

and s_n, $\bar{r}_n$ and $\bar{f}_n$ are the approximations to $\bar{u}(t_n)$, $\bar{r}(t_n)$ and $\bar{f}(t_n)$, respectively. Derivations of the second and third equations in equation (5) are exactly the same as the procedure derived for the Newmark explicit method[16] except for the changes of parameters from acceleration, velocity and displacement to velocity, displacement and the integral of displacement, respectively. The basic properties in stability, accuracy, convergence, dissipation, dispersion and overshoot behaviour of this formulation are found to be exactly the same as the Newmark explicit method. For free vibrations, the relative period error versus $\Delta t/T$ for the two integration algorithms are shown in Figure 1 where T is the natural period of the vibration system and $\bar{T}$ is the computed period corresponding to T in a numerical solution. These two integration methods possessing exactly the same period distortion error for the case of free vibration response are manifested by this figure.

In this form, the difficulty caused by the rapid variation of external excitation can be overcome since the time integral of external force is proposed to avoid the direct use of the external forces which is generally used by the currently available step-by-step integration methods. It is worth noting that the time integral of external force can be easily obtained accurately even for a set of measured data. In this manner, the favourable smoothing effect of time integration eliminates the need of using a very small time step for

simulating the rapid changes of dynamic loading. Meanwhile, the time integral of the restoring force can effectively eliminate the linearization errors since the restoring force can be measured many times or almost continuously in a time step during a pseudodynamic test. Based on this concept, it seems that this technique can be incorporated into any other direct integration methods without encountering any difficulty.

Using this approach, if the second equation in equation (5) is multiplied by the tangent stiffness k and the resulting terms $k(\Delta s_{n+1})$ and $k(\Delta t)d_n$ can be replaced by $\Delta\bar{r}_{n+1}$ and $(\Delta t)r_n$, respectively, one has

$$\Delta\bar{r}_{n+1} = (\Delta t)r_n + \frac{k}{2}(\Delta t)^2 v_n \tag{7}$$

This expression is exactly the same as the second equation of equation (5) for a linear system while it may be different for a non-linear system if the resistance is non-linearly varied during the time step. However, it will still provide a reliable approximation if a small enough time step is used. It is worth noting that the second equation of equation (5) itself is also an approximation equation which is a Taylor expansion of displacement without remainder and is limited to the second-order time derivative. This implementation is beneficial for pseudodynamic tests since the restoring force r_n can be measured experimentally. Substituting equation (7) and the third equation of equation (5) into the first equation of equation (5), the following expression can be obtained:

$$\Delta v_{n+1} = \left(m + \frac{\Delta t}{2}c\right)^{-1}\left(\Delta\bar{f}_{n+1} - c\Delta t v_n - \Delta t r_n - \frac{k}{2}(\Delta t)^2 v_n\right) \tag{8}$$

Thus, the displacement increment can be computed by the substitution of equation (8) into the third equation of equation (5) and is found to be

$$\Delta d_{n+1} = \Delta t v_n + \frac{\Delta t}{2}\left(m + \frac{\Delta t}{2}c\right)^{-1}\left(\Delta\bar{f}_{n+1} - c\Delta t v_n - \Delta t r_n - \frac{k}{2}(\Delta t)^2 v_n\right) \tag{9}$$

In this formulation, the use of the tangent stiffness k is required and it is generally updated for each time step in order to approximate the non-linear behaviour of the system. It is clear that this procedure is not feasible for a non-linear system since the actual tangent stiffness for each time step cannot be determined by the pseudodynamic test. A possible alternative is to replace the tangent stiffness k with the initial stiffness k_0. Thus, equation (9) becomes

$$\Delta d_{n+1} = \Delta t v_n + \frac{\Delta t}{2}\left(m + \frac{\Delta t}{2}c\right)^{-1}\left(\Delta\bar{f}_{n+1} - c\Delta t v_n - \Delta t r_n - \frac{k_0}{2}(\Delta t)^2 v_n\right) \tag{10}$$

At first glance, it seems that the use of the initial stiffness to take place of the tangent stiffness for the whole pseudodynamic test is incorrect for a non-linear system since the tangent stiffness may vary for each time step. However, this will not cause significant errors since it is only a high-order term when compared with the rest terms.

The detailed implementation of the integral form of Newmark explicit method starting at the beginning of $(n+1)$th step can be summarized as

(1) Compute the displacement increment for the next step Δd_{n+1}, using equation (10).
(2) Impose the computed displacement increment on the specimen.
(3) Measure the restoring force several times or continuously from the beginning to the end of this step and then integrate it to obtain $\bar{r}_{n+1}$.
(4) Compute velocity v_{n+1}, using the first equation of equation (5) using the available $\Delta\bar{r}_{n+1}$ and Δd_{n+1}.
(5) Repeat.

In this test procedure, the integrals of the external forces and the restoring forces can be computed by using the trapezoidal rule. The time integration of external forces may be computed before the test and then stored as an earthquake input record.

COMPARISONS OF INTEGRATION METHODS

The major difference between the Newmark explicit method and the proposed integral form is the governing equations. The original form of equations of motion are used by the Newmark explicit method while the equations of motion are integrated with respect to time once for the integral form of Newmark explicit method.

Capability to capture the rapid changes of dynamic loading

In equation (3), for the Newmark explicit method, the external force is used in the formulation. This may lead to the use of a very small time step to capture the rapid changes of dynamic loading for a high-frequency content record. On the other hand, for the integral form of Newmark explicit method as pressed in equation (5), the external force is integrated with respect to time. The rapid changes of dynamic loading can be smoothed out by the time-integration procedure. The extensive study of this time-integration effect can be found in References[1–3].

Based on specific required accuracy, the upper bound of $\Delta t/T$ as a function of T/T^* for accurate integration of linear systems subjected to a harmonically varying load has been developed by Chang[2,3] for the Newmark explicit method and its integral form. The symbol T represents the natural period of the vibration system as defined before while T^* stands for the period of the harmonic load applied. Since the errors introduced by direct integration methods are often manifested in the forms of period distortion and amplitude changes, we would like to establish this upper bound so that the period distortion is not greater than 1 per cent and the amplitude growth or decay is less than 3% after 10 times of the period of the single-degree-of-freedom system. These upper bounds for the system with a natural period of $T = 1$ s are shown in Figure 2. It is apparent that the integral form of Newmark explicit method possesses a much better capability in capturing the rapid changes of dynamic loading than the Newmark explicit method does for $T/T^* > 3$, while it is almost the same for both integration methods as $T/T^* < 3$. Although these studies are based on harmonically varying loads, these guidelines believe to be indicative for earthquake excitations since the ground accelerations can be considered as a combination of harmonic loads.

Linearization errors

During the pseudodynamic test using a step-by-step integration method, the non-linear behaviour of the system is taken into account by evaluating new properties corresponding to the current deformed state at the start of each time step. If the non-linear behaviour is of the elasto-plastic type, changes in stiffness occur only upon yielding or unloading. Since the new properties are updated at the beginning of each time step, the abrupt changes of material properties might occur during a time step and cannot be captured accurately by using the old properties for the entire time step. This situation is illustrated in Figure 3 where R_n is used to compute the responses of $(n + 1)$th time step. Obviously, the use of R_n for the entire time step is incorrect since it should be replaced by R_y as the structure yields. For this case, the triangular area as shown in the figure is the error introduced in this time step. Such an approximation might lead to a considerable error in a pseudodynamic test. Depending on how large the stiffness change and the size of time step are, the linearization may introduce significant errors and lead to poor test results. It is very important to note that the geometric non-linearity or the smooth variation of material properties may also introduce linearization errors.

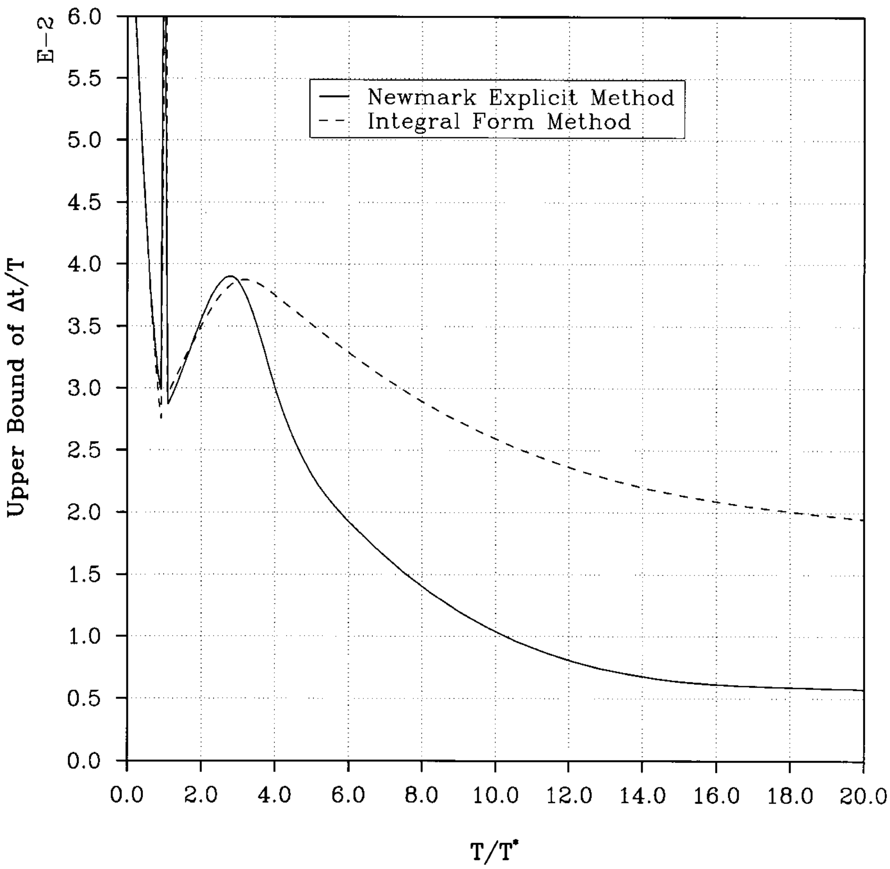

Figure 2. Upper bound of $\Delta t/T$ with different values of T/T^*

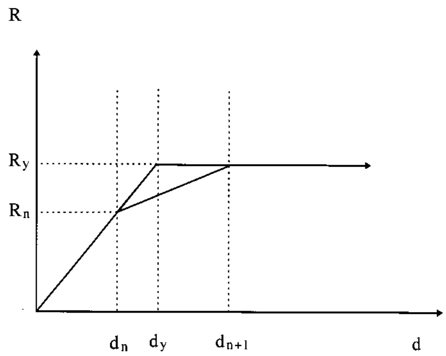

Figure 3. Force–displacement relationship

Unlike the Newmark explicit method, the adverse linearization errors can be automatically eliminated if the integral form of Newmark explicit method is applied. In step (3) of the detailed implementation for the integral form of Newmark explicit method, an original time step Δt can be subdivided into j sub-steps:

$$\Delta\tau = \frac{\Delta t}{j} \tag{11}$$

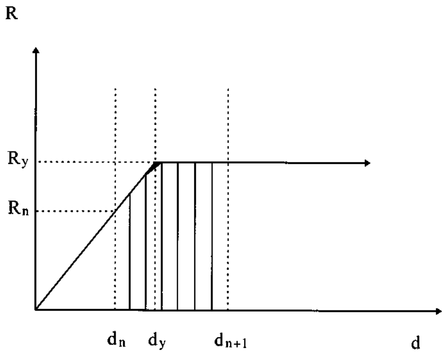

Figure 4. Force–displacement relationship

where $\Delta\tau$ is the sub-step size. The trapezoidal rule can be used to integrate the measured restoring force. It can be expressed as

$$\bar{r}_{n+1} = \bar{r}_n + \sum_{j=0}^{j} \left(\frac{\Delta\tau}{2}\right)(r_i + r_{i+1}) \tag{12}$$

where r_0 is equal to r_n and r_j is equal to r_{n+1} (see Figure (4)). The value of each r_i can be measured experimentally during the test and the variation of the restoring force within a sub-step is assumed to be linear. Obviously, the linearization errors caused by the rapid variation of the resistance can be significantly reduced or almost entirely eliminated. Figure 4 schematically demonstrate this technique. For an elasto-plastic system, the shaded area is the only error introduced. This error is not significant and it can become smaller if a small refinement is adopted. It is clear that linearization errors also indicate energy changes (energy addition or energy dissipation) of the system in a pseudodynamic test. These energy effects can have a significant influence on the response of the system. Spurious amplitude growth or even numerical instability may occur if energy is numerically added into the system while an energy dissipation effect might result in an excessive amplitude decay.

Computational efficiency

In the pseudodynamic test method, the computational cost is generally not a major concern in practical applications since only very limited degree of freedoms of the specimen will be tested. However, the comparisons of the computational efficiency for the two integration methods are informative in the step-by-step dynamic analysis. It can be easily found that the three basic unknowns s_{n+1}, d_{n+1}, and v_{n+1} with $\bar{f}_{n+1}$ to take place of the unknowns d_{n+1}, v_{n+1}, and a_{n+1} with f_{n+1} are the major change between the Newmark explicit method and its integral form. Other than this, the computational procedures for the two integration methods are almost the same.

For linear systems there will be no extra computational effort required for the integral form of Newmark explicit method in each time step except the cost to compute the time integral of external forces. This computation takes very little time and only needs to be performed once. An additional computational effort required for a non-linear system in each time step is to compute the time integral of restoring force. However, this can be easily done by using the trapezoidal rule which only involves very limited computations. It has been verified that the incorporation of the time-integration technique to direct integration methods may be computationally more efficient in dealing with the rapid changes of dynamic loading and the variation of resistance. Detailed investigations regarding to the computational efficiency for the applications of this time-integration technique to step-by-step integration methods in performing non-linear dynamic analysis can be found in References 1–3.

ERROR PROPAGATION ANALYSIS

For error propagation analysis, it is desirable to express the Newmark explicit method and its integral form each in a recursive matrix form for a single-degree-of-freedom system. The detailed derivations of the following expressions are exactly the same except the changes of the dependent variables and the detailed derivation for the Newmark explicit method can be found in References 4 and 5.

$$\begin{aligned} \mathbf{X}_{n+1} &= \mathbf{A}\mathbf{X}_n + \mathbf{L}\mathbf{f}_{n+1} = \mathbf{A}^{(n+1)}\mathbf{X}_0 + \sum_{i=0}^{n} \mathbf{A}^{(n-i)}\mathbf{L}\mathbf{f}_{i+1} \\ \bar{\mathbf{X}}_{n+1} &= \mathbf{A}\bar{\mathbf{X}}_n + \mathbf{L}\bar{\mathbf{f}}_{n+1} = \mathbf{A}^{(n+1)}\bar{\mathbf{X}}_0 + \sum_{i=0}^{n} \mathbf{A}^{(n-i)}\mathbf{L}\bar{\mathbf{f}}_{i+1} \end{aligned} \tag{13}$$

in which $\mathbf{X}_n = [d_n, (\Delta t) v_n, (\Delta t)^2 a_n]^{\mathrm{T}}$ is defined for the Newmark explicit method and $\bar{\mathbf{X}}_n = [s_n, (\Delta t) d_n, (\Delta t)^2 v_n]^{\mathrm{T}}$ is introduced for its integral form. The load factor $\mathbf{L}$ and the amplification matrix $\mathbf{A}$ are found to be

$$\mathbf{L} = \left(\frac{\Omega^2}{2k}\right) \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \tag{14}$$

and

$$\mathbf{A} = \left(\frac{1}{4}\right) \begin{bmatrix} 4 & 4 & 2 \\ -2\Omega^2 & 4-2\Omega^2 & 2-\Omega^2 \\ -4\Omega^2 & -4\Omega^2 & -2\Omega^2 \end{bmatrix} \tag{15}$$

in which $\Omega = \omega(\Delta t)$ and ω is the natural frequency of the single-degree-of-freedom system.

In a pseudodynamic test, experimental errors are unavoidable and are introduced in each step. Furthermore, these errors are carried over to the rest of the test. Consequently, the test results in any step depend on an interaction from every previous step. It is somewhat difficult to impose the target displacement on the specimen perfectly due to the displacement control error. In addition, the actual restoring force developed in the specimen may not be so perfectly measured and interpreted by the computer with errors. These control and measurement errors possess a cumulative effect from the beginning to the end. As a result, the error propagation effect is that incorrect imposed displacements lead to incorrect force feedback, and the errors in restoring forces further result in incorrect displacements subsequently computed and imposed in the next steps.

The following notation is defined for the convenience of subsequent derivations and discussions.

d_n = exact numerically computed displacement at step n without errors
r_n = exact numerically computed restoring force at step n without errors
d_n^e = exact displacement at step n, including the effects of errors at previous steps
r_n^e = exact restoring force at step n, including the effects of errors at previous steps
d_n^{a} = actual displacement at step n, including the effects of previous errors and errors introduced at the current step
r_n^{a} = actual restoring force at step n, including the effects of previous errors and errors introduced at the current step
e_n^{d} = displacement error introduced at step n
e_n^{r} = force error introduced at step n

In pseudodynamic testing, the displacement control error and the force measurement error are the major sources of the displacement and force errors. In general, the measurement errors are less than the control errors. Using the above notation, the following relationships can be defined:

$$
\begin{aligned}
d^{a}_{n+1} &= d^{e}_{n+1} + e^{d}_{n+1} \\
r^{a}_{n+1} &= r^{e}_{n+1} + e^{r}_{n+1} \\
\bar{r}^{a}_{n+1} &= \bar{r}^{e}_{n+1} + e^{\bar{r}}_{n+1}
\end{aligned}
\tag{16}
$$

where the force feedback error e^{r}_{n+1} and the integral of the force feedback error $e^{\bar{r}}_{n+1}$ can be replaced by

$$
\begin{aligned}
e^{r}_{n+1} &= k e^{rd}_{n+1} \\
e^{\bar{r}}_{n+1} &= k(\Delta t) e^{rd}_{n+1}
\end{aligned}
\tag{17}
$$

in which e^{rd}_{n+1} is the amount of displacement error corresponding to e^{r}_{n+1} for a linear single-degree-of-freedom system. The second equation in equation (17) implies that the displacement error introduced in each time step is assumed to be constant during the time step for the integral form of Newmark explicit method. The recursive matrix forms as shown in equation (13) can be reformulated based on the actual restoring forces and displacements:

$$
\begin{aligned}
\mathbf{X}^{e}_{n+1} &= \mathbf{A}\mathbf{X}^{e}_{n} + \mathbf{L} f_{n+1} + \mathbf{M} e^{d}_{n} - \mathbf{N} e^{rd}_{n+1} \\
\bar{\mathbf{X}}^{e}_{n+1} &= \mathbf{A}\bar{\mathbf{X}}^{e}_{n} + \mathbf{L} \bar{f}_{n+1} + \bar{\mathbf{M}}(\Delta t) e^{d}_{n} - \bar{\mathbf{N}}(\Delta t) e^{rd}_{n+1}
\end{aligned}
\tag{18}
$$

in which $\mathbf{X}^{e}_{n} = [d^{e}_{n}, (\Delta t) v^{e}_{n}, (\Delta t)^2 a^{e}_{n}]$ and $\bar{\mathbf{X}}^{e}_{n} = [s^{e}_{n}, (\Delta t) d^{e}_{n}, (\Delta t)^2 v^{e}_{n}]$. The vectors $\mathbf{M}$, $\bar{\mathbf{M}}$, $\mathbf{N}$ and $\bar{\mathbf{N}}$ are

$$
\mathbf{M} = \begin{bmatrix} 1 \\ -\Omega^2/2 \\ -\Omega^2 \end{bmatrix}, \qquad
\bar{\mathbf{M}} = \begin{bmatrix} 1 \\ 1-\Omega^2/2 \\ -\Omega^2 \end{bmatrix}, \qquad
\mathbf{N} = \begin{bmatrix} 0 \\ \Omega^2/2 \\ \Omega^2 \end{bmatrix}, \qquad
\bar{\mathbf{N}} = \begin{bmatrix} 0 \\ 0 \\ \Omega^2/2 \end{bmatrix}
\tag{19}
$$

After the subtraction of equation (13) from equation (18), we have

$$
\begin{aligned}
\varepsilon_{n+1} &= \sum_{i=0}^{n} \mathbf{A}^{(n-i)} \mathbf{M} e^{d}_{i} - \sum_{i=0}^{n} \mathbf{A}^{(n-i)} \mathbf{N} e^{rd}_{i+1} \\
\bar{\varepsilon}_{n+1} &= \sum_{i=0}^{n} \mathbf{A}^{(n-i)} \bar{\mathbf{M}} (\Delta t) e^{d}_{i} - \sum_{i=0}^{n} \mathbf{A}^{(n-i)} \bar{\mathbf{N}} (\Delta t) e^{rd}_{i+1}
\end{aligned}
\tag{20}
$$

where $\varepsilon_{n+1} = \mathbf{X}^{e}_{n+1} - \mathbf{X}_{n+1}$ and $\bar{\varepsilon}_{n+1} = \bar{\mathbf{X}}^{e}_{n+1} - \bar{\mathbf{X}}_{n+1}$. In these derivations, $\varepsilon_0 = \bar{\varepsilon}_0 = \mathbf{0}$ is taken since no errors introduced at the beginning of the test are assumed. The first term on the right-hand side of the equation is the cumulative error due to the displacement feedback errors and the second term is due to the force feedback errors.

Substituting the explicit expressions for $\mathbf{A}$, $\mathbf{M}$, $\bar{\mathbf{M}}$, $\mathbf{N}$ and $\bar{\mathbf{N}}$ into equation (20) and applying the spectral decomposition technique,[18] the equations for describing the error propagation effects of the Newmark

explicit method and its integral form can be derived. Thus, the representations of the cumulative errors for the Newmark explicit method and its integral form can be expressed as

$$
\begin{aligned}
(\varepsilon_{n+1})_1 &= C_1 \sum_{i=0}^{n} \cos[(n-i+\tfrac{1}{2})\bar{\Omega}]\, e_i^{\mathrm{d}} - D_1 \sum_{i=0}^{n-1} \sin[(n-i)\bar{\Omega}]\, e_{i+1}^{\mathrm{rd}} - E_1 e_{n+1}^{\mathrm{rd}} \\
(\varepsilon_{n+1})_2 &= C_2 \sum_{i=0}^{n} \sin[(n-i+\tfrac{1}{2})\bar{\Omega}]\, e_i^{\mathrm{d}} - D_2 \sum_{i=0}^{n-1} \cos[(n-i)\bar{\Omega}]\, e_{i+1}^{\mathrm{rd}} - E_2 e_{n+1}^{\mathrm{rd}} \\
(\varepsilon_{n+1})_3 &= C_3 \sum_{i=0}^{n} \cos[(n-i+\tfrac{1}{2})\bar{\Omega}]\, e_i^{\mathrm{d}} - D_3 \sum_{i=0}^{n-1} \sin[(n-i)\bar{\Omega}]\, e_{i+1}^{\mathrm{rd}} - E_3 e_{n+1}^{\mathrm{rd}} \\
(\bar{\varepsilon}_{n+1})_1 &= \bar{C}_1 \sum_{i=0}^{n} \sin[(n-i+1)\bar{\Omega}]\,(\Delta t) e_i^{\mathrm{d}} - \bar{D}_1 \sum_{i=0}^{n-1} \sin[(n-i)\bar{\Omega}](\Delta t) e_{i+1}^{\mathrm{rd}} - \bar{E}_1 (\Delta t) e_{n+1}^{\mathrm{rd}} \\
(\bar{\varepsilon}_{n+1})_2 &= \bar{C}_2 \sum_{i=0}^{n} \cos[(n-i+1)\bar{\Omega}]\,(\Delta t) e_i^{\mathrm{d}} - \bar{D}_2 \sum_{i=0}^{n-1} \cos[(n-i)\bar{\Omega}](\Delta t) e_{i+1}^{\mathrm{rd}} - \bar{E}_2 (\Delta t) e_{n+1}^{\mathrm{rd}} \\
(\bar{\varepsilon}_{n+1})_3 &= \bar{C}_3 \sum_{i=0}^{n} \sin[(n-i+1)\bar{\Omega}]\,(\Delta t) e_i^{\mathrm{d}} - \bar{D}_3 \sum_{i=0}^{n-1} \sin[(n-i)\bar{\Omega}](\Delta t) e_{i+1}^{\mathrm{rd}} - \bar{E}_3 (\Delta t) e_{n+1}^{\mathrm{rd}}
\end{aligned}
\tag{21}
$$

The symbols $(\varepsilon_{n+1})_j$ and $(\bar{\varepsilon}_{n+1})_j$ where $j = 1, 2$ and 3 are defined as follows:

$$
\begin{aligned}
(\varepsilon_{n+1})_1 &= d_{n+1}^{\mathrm{e}} - d_{n+1} = e_{n+1}^{\mathrm{d}} \\
(\varepsilon_{n+1})_2 &= (\Delta t)(v_{n+1}^{\mathrm{e}} - v_{n+1}) \\
(\varepsilon_{n+1})_3 &= (\Delta t)^2 (a_{n+1}^{\mathrm{e}} - a_{n+1})
\end{aligned}
\tag{22}
$$

and

$$
\begin{aligned}
(\bar{\varepsilon}_{n+1})_1 &= s_{n+1}^{\mathrm{e}} - s_{n+1} \\
(\bar{\varepsilon}_{n+1})_2 &= (\Delta t)(d_{n+1}^{\mathrm{e}} - d_{n+1}) = (\Delta t) e_{n+1}^{\mathrm{d}} \\
(\bar{\varepsilon}_{n+1})_3 &= (\Delta t)^2 (v_{n+1}^{\mathrm{e}} - v_{n+1})
\end{aligned}
\tag{23}
$$

The error amplification factors C_j, D_j, E_j, $\bar{C}_j$, $\bar{D}_j$ and $\bar{E}_j$ for the Newmark explicit method and its integral form are summarized in Table I:

Using these error amplification factors, it can be easily shown that for the Newmark explicit method, the error propagation effect of the displacement or acceleration becomes infinite while it remains bounded for the velocity as the stability limit is approached. It is clear that the error amplification factors C_1 and D_1 as well as C_3 and D_3 which correspond to the displacement and acceleration error propagation effects, respectively, will tend to infinity as $\Omega \to 2$. On the other hand, $C_2 = -\Omega$, $D_2 = \Omega^2$ and $E_2 = \Omega^2/2$ indicate that the error propagation effect for velocity remains bounded as $\Omega \to 2$. Similarly, for the integral form of Newmark explicit method, it also can be shown that this effect will not grow out of bound for the displacement while it blows up for the time integral of displacement and velocity as $\Omega \to 2$. It is very interesting to find why the displacement can remain bounded while the velocity can grow without limit for the integral form of Newmark explicit method. If $c = 2\xi\omega m$ is introduced into equation (9). Then, equation (9), which will be ill-conditioned as $v_n \to \infty$, can be rewritten as

$$
\Delta d_{n+1} = \left(\frac{1 - \frac{1}{4}\Omega^2}{1 + \xi\Omega} \right) (\Delta t) v_n + \frac{1}{2(1+\xi\Omega)} \left[\left(\frac{\Delta t}{m} \right) \Delta \bar{f}_{n+1} - \Omega^2 d_n \right]
\tag{24}
$$

Table I. Error amplification factors

Method	Newmark explicit method			Integral form method		
	C_j	D_j	E_j	$\bar{C}_j$	$\bar{D}_j$	$\bar{E}_j$
$j=1$	$\dfrac{1}{\sqrt{1-\Omega^2/4}}$	$\dfrac{\Omega}{\sqrt{1-\Omega^2/4}}$	0	$\dfrac{1}{\Omega\sqrt{1-\Omega^2/4}}$	$\dfrac{\frac{1}{2}\Omega}{\sqrt{1-\Omega^2/4}}$	0
$j=2$	$-\Omega$	Ω^2	$\dfrac{\Omega^2}{2}$	1	$\dfrac{\Omega^2}{2}$	0
$j=3$	$\dfrac{-\Omega^2}{\sqrt{1-\Omega^2/4}}$	$\dfrac{-\Omega^3}{\sqrt{1-\Omega^2/4}}$	Ω^2	$\dfrac{-\Omega}{\sqrt{1-\Omega^2/4}}$	$\dfrac{-\frac{1}{2}\Omega^3}{\sqrt{1-\Omega^2/}}$	Ω^2

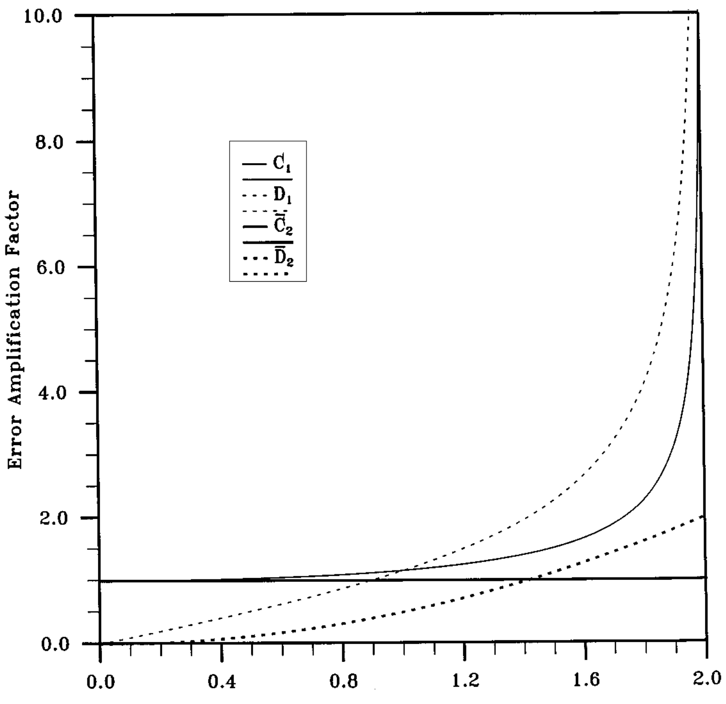

Figure 5. Comparisons of error amplification factors

In this expression, it is clear that the velocity term will be diminished as the stability limit 2 is approached. Thus, the displacement can remain bounded even though the unbounded velocity is occurred. It should be mentioned that the displacement is of great importance in a conventional pseudodynamic test while the acceleration, velocity and time integral of displacement are of no interest. This indicates that the integral form of Newmark explicit method possesses better error propagation properties as the stability limit is approached.

To gain a better understanding of the error propagation effect in displacement for both integration methods, it is very useful to investigate the variations of the error amplification factors with respect to Ω. The values of C_1 and D_1 as well as $\bar{C}_2$ and $\bar{D}_2$ are plotted against Ω in Figure 5. It is evident from this figure that the integral form of Newmark explicit method is a better scheme in terms of error propagation since the error amplification factors $\bar{C}_2$ and $\bar{D}_2$ are less than C_1 and D_1, respectively, for any value of Ω within the stability limit 2.

NUMERICAL EXAMPLES

In performing a pseudodynamic test, the actuator response might consistently undershoot or overshoot the command signal. This type of error causes the greatest propagation problem,[4] and thus is chosen to be simulated herein. To illustrate the error propagation characteristics for these two integration methods, two linearly elastic single-degree-of-freedom systems subjected to the NS component of the 1940 El Centro earthquake ground motion with the peak ground acceleration scaled to $0{\cdot}35\,g$ are considered. The undershooting error for each time step is introduced in the displacement for both the Newmark explicit method and its integral form as shown in Figure 6.

Example 1

In this example, a system with mass $m = 10$ kg and stiffness $k = 4 \times 10^3$ N/m is considered and the undershooting error for each time step is assumed to be 0·1 mm. All the numerical solutions are obtained by using the time step of 0·02 s. The numerical results obtained from the Newmark explicit method without the presence of undershooting errors are considered as 'exact' solutions for comparison purposes. The results are shown in Figure 6. This figure contains three plots. The top plot shows the displacement response time history, the middle one depicts the time history of undershooting errors, and the cumulative errors are described in the bottom plot. It is manifested by the bottom plot that the integral form of Newmark explicit method possesses a less-error propagation effect when compared with the Newmark explicit method since it shows less cumulative errors. This also can be explained by the results indicated in Figure 5. In this figure for the case of $\Omega = 0{\cdot}4$ in which $\omega = 20$ rad/s and $\Delta t = 0{\cdot}02$ s, the error amplification factors C_1 and D_1 for the Newmark explicit method are 1·021 and 0·408 while the error amplification factors $\bar{C}_2$ and $\bar{D}_2$ for its integral form are 1·000 and 0·080.

Example 2

A single-degree-of-freedom system, whose mass is $m = 10$ kg and its stiffness is $k = 9{\cdot}025 \times 10^4$ N/m, is used to demonstrate the error propagation characteristics for the Newmark explicit method and its integral form as the value of $\omega(\Delta t)$ tends to the stability limit. The natural frequency of this system is found to be 95 rad/s. The step sizes used in all computations are chosen to be 0·02 s. Thus, these lead to $\Omega = 1{\cdot}9$. The undershooting errors are simulated by simply assuming that the actuator response will consistently lag behind the computed displacement increment with 0·1 mm for each time step.

Displacement responses of the system using the Newmark explicit method and its integral form with the presence of undershooting errors and those obtained from the Newmark explicit method without any undershooting errors are plotted in Figure 7. The period for each displacement time history is significantly distorted since the value of Ω is equal to 1·9. Therefore, all the numerical results are incorrect. However, these numerical results are good enough to show the characteristics of the error propagation effect for both integration methods. It is clear that the displacement responses obtained from the Newmark explicit method with the presence of the simulated undershooting errors blow up very rapidly while those obtained from the integral form of Newmark explicit method remain bounded. The result of these numerical simulations are in good agreement with the previous theoretical studies.

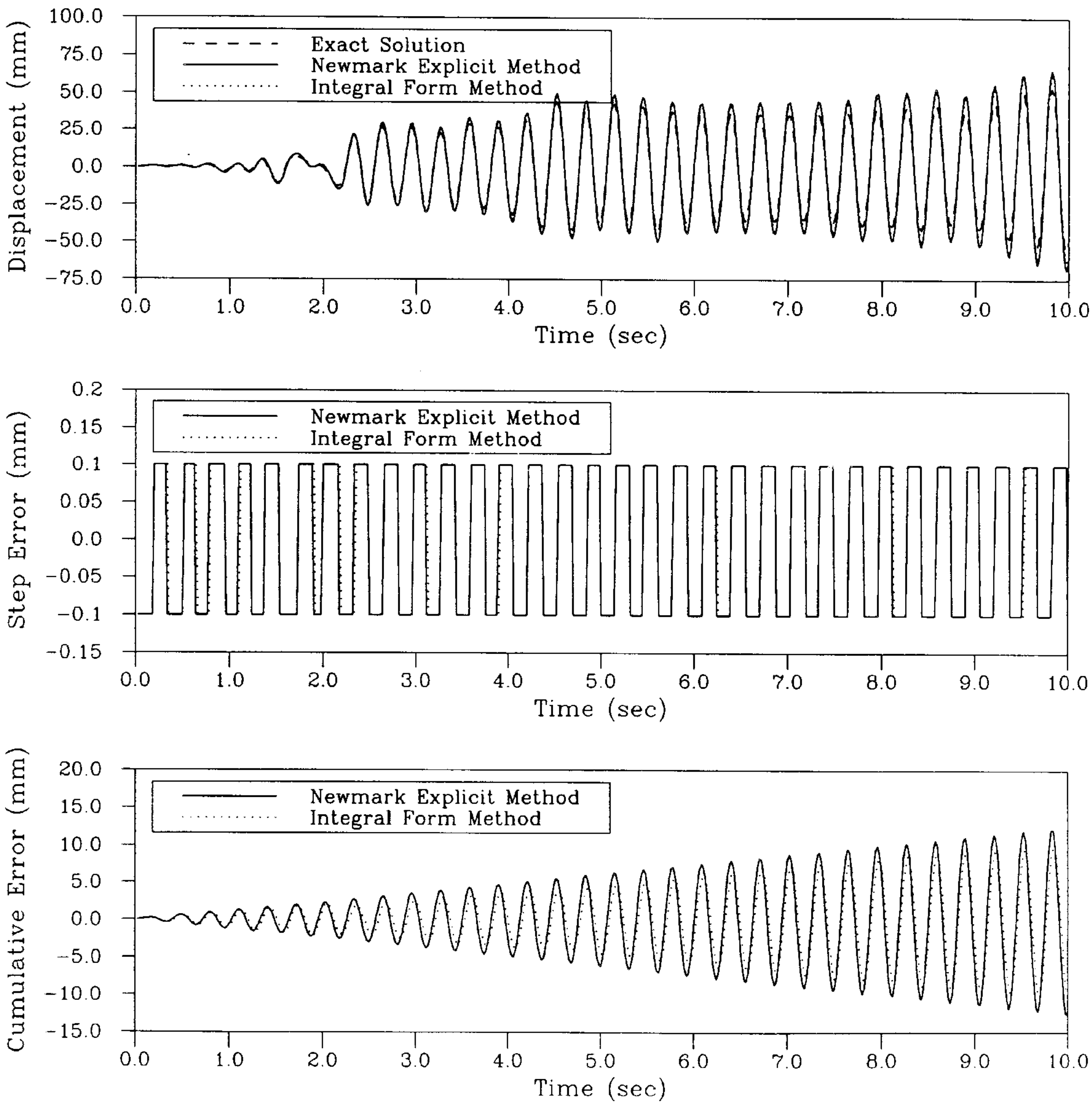

Figure 6. Response to El Centro Record with undershooting error of 0·1 mm

VERIFICATION TESTS

In order to investigate the smoothing effectiveness of the proposed time-integration procedure in the pseudodynamic tests, a series of comparable tests were performed. Both the original Newmark explicit method and the proposed integral formulation are examined in these tests. In this study, pseudodynamic tests of the single-degree-of-freedom system are conducted in order to investigate (1) the performance of the proposed integral approach in capturing the dynamic loading, and (2) its effectiveness in eliminating the adverse linearization errors.

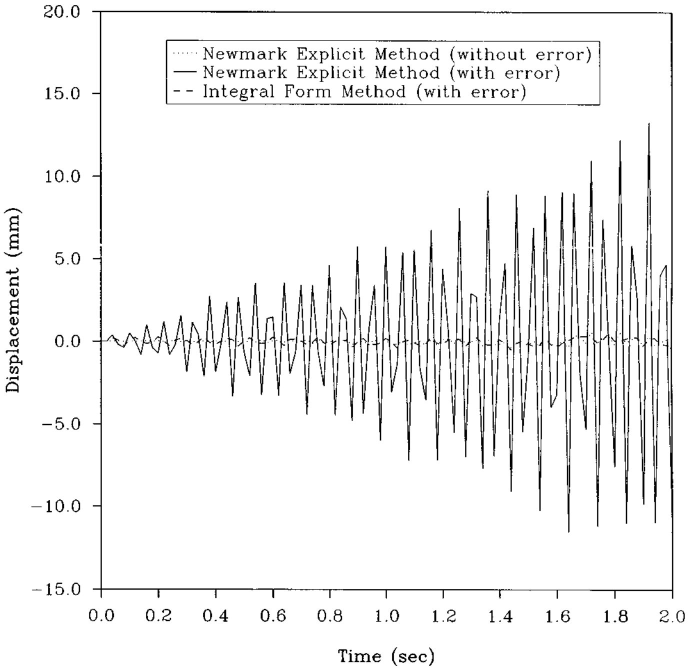

Figure 7. Response to El Centro Record with undershooting error of 0·1 mm

It is automatically assumed that all the step sizes used in the following tests are chosen so small that each value of $\Delta t/T < 0{\cdot}05$. This implies that all the relative period errors are less than 0·5 per cent (see Figure 1) and, thus, very accurate solutions can be obtained by using the Newmark explicit method or its integral form based on accuracy consideration. Therefore, the possible causes of inaccurate results may be due to the rapid changes of dynamic loading and/or the rapid variation of resistance but not due to accuracy consideration in all the following examples.

Test program

The Triangular-Plate Added Damping and Stiffness (TADAS) devices are chosen for the test specimens as the device can yield at a very small deformation and undergo extremely large inelastic strain reversals.[19] The fabrication details of a TADAS device are given in Figure 8. The cyclic force versus deformation relationships of a typical TADAS device are shown in Figure 9. Past experimental tests have confirmed that properly designed TADAS devices can absorb a large amount of hysteresis energy thereby reducing the structural responses during severe earthquakes.[19,20] The mechanical property of a TADAS device is highly predictable and has been documented in the References.[19,21]

The dimensions and the mechanical properties of three TADAS device specimens are given in Table II, where t, h, B, F_Y and N stand for the thickness, the height, the width, the tensile yield strength and the number of triangular plates, respectively. In addition, K, V_Y and D_Y are the lateral elastic stiffness, yield strength and yield displacement of the device, respectively. Specimens A1 and A2 were tested into non-linear responses and Specimen B was tested within the linear elastic range only. During the test, the base plate of the TADAS specimen was anchored on a rigid steel beam, and the lateral displacements were imposed by a servo-controlled hydraulic actuator through a strut connecting the circular bars at the end of each triangular plate (Figures 8 and 10).

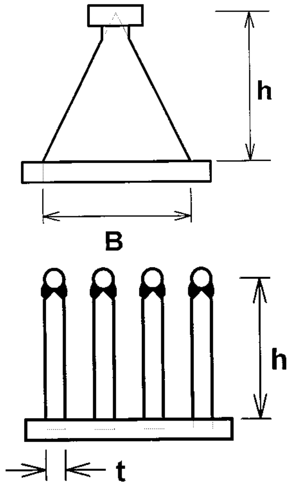

Figure 8. The details for a TADAS device

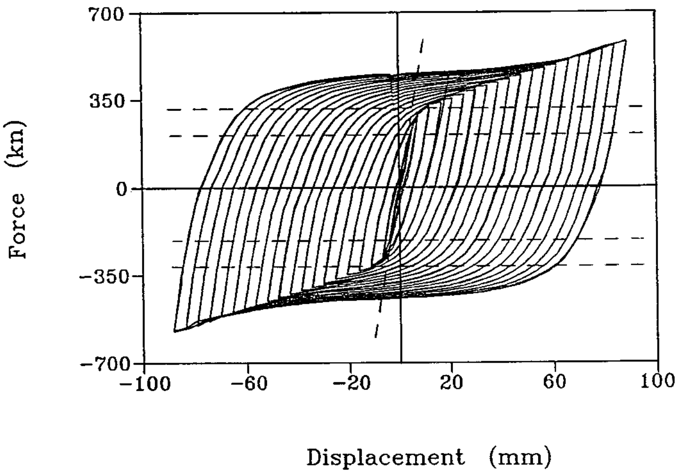

Figure 9. A typical hysteretic loop for a TADAS device

Table II. Properties of TADAS device specimens

Specimen	N	t (mm)	h (mm)	B (mm)	F_Y (Mpa)	K (kN/mm)	V_Y (kN)	D_Y (mm)
A1 & A2	4	35	320	180	253	24·45	115	4·68
B	4	25	320	180	259	9·01	59	6·55

A MTS actuator and a MTS458-10 controller was used for the servo-control algorithms. The actuator displacement ranges were calibrated to $\pm$ 25 and $\pm$ 125 mm for linear and non-linear tests, respectively. In addition to actuator's built-in displacement transducer, external displacement gages of ranges $\pm$ 5 and $\pm$ 100 mm were also applied for the linear and non-linear tests, respectively. A 16-bit A/D converter was

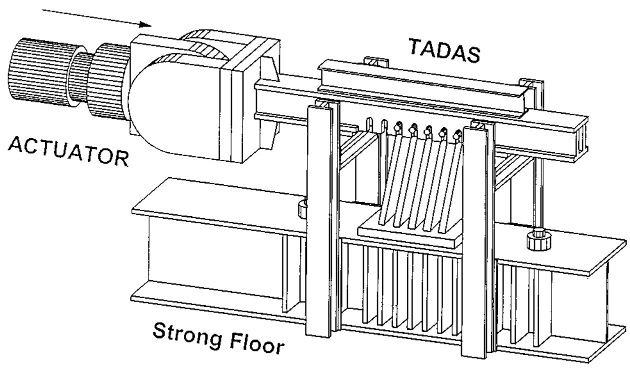

Figure 10. The test set-up for a pseudodynamic test

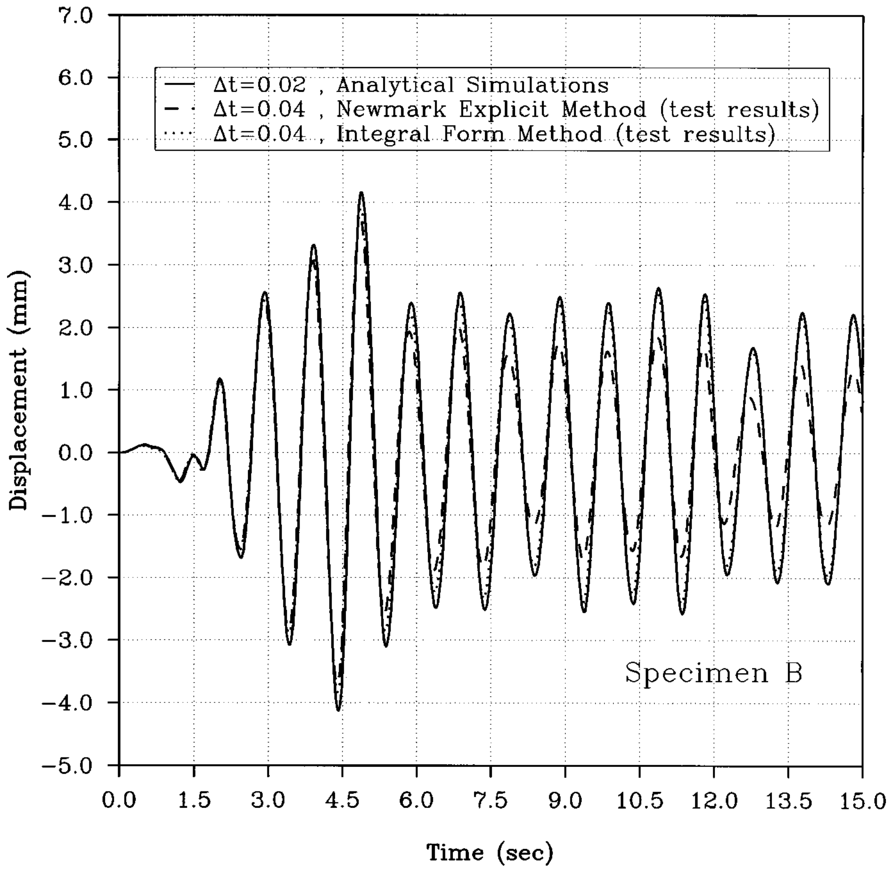

Figure 11. Pseudodynamic response to 0·007 g El Centro Record

used in the interface and the D/A converter was 18 bit. The displacement tolerances were specified to be 0·0125 and 0·0625 mm (0·05 per cent of the actuator displacement range) for elastic and inelastic tests, respectively. During the test, the actuator movement was continuously monitored as it led the test structure to the target displacement. If an overshoot was detected, a command was sent to stop the actuator

immediately. On the other hand, if the actuator lagged behind the target displacement, it kept going until the amount of undershoot was within the displacement tolerance.

Test results for linear systems

In order to illustrate that the integral form of Newmark explicit method is more effective in capturing the rapid changes of dynamic loading, a series of linear elastic pseudodynamic tests were performed. Specimen B was subjected to the El Centro record in which the peak ground acceleration was scaled to 0·007 *g*. Here, the use of an earthquake record with low acceleration is because that the TADAS device specimen possesses a very small linear elastic range. In this simple single-degree-of-freedom system, the assumed mass was 0·228 kg and the measured stiffness was 9·01 kN/mm. The resulting natural period for the system is 1 s.

Both the analytical simulations and the test results are given in Figure 11. The analytical simulations were obtained by using the Newmark explicit method with a time step of 0·02 s and are assumed to be the "exact" responses for comparison purposes. The test results show that the Newmark explicit method cannot provide reliable results for a time step of 0·04 s while very accurate results can be obtained if using the integral form Newmark explicit method with a time step of 0·04 s. Apparently, this is due to the superiority of the integral form of Newmark explicit method in capturing the rapid changes of dynamic loading.

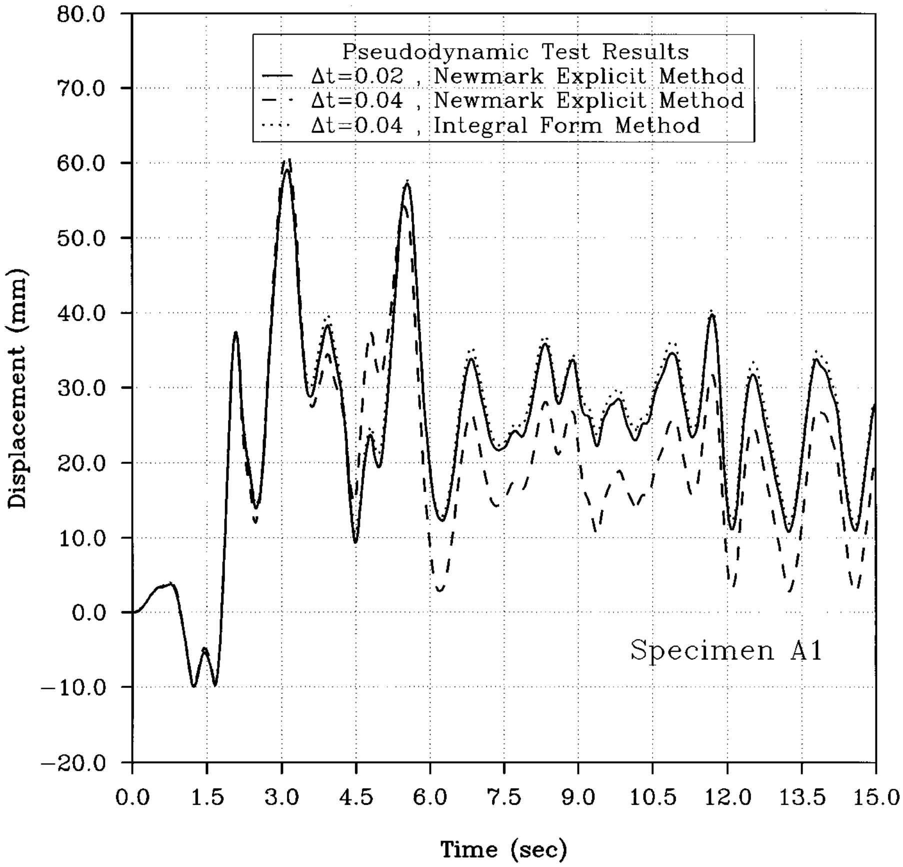

Figure 12. Pseudodynamic response to 0·15 *g* El Centro Record

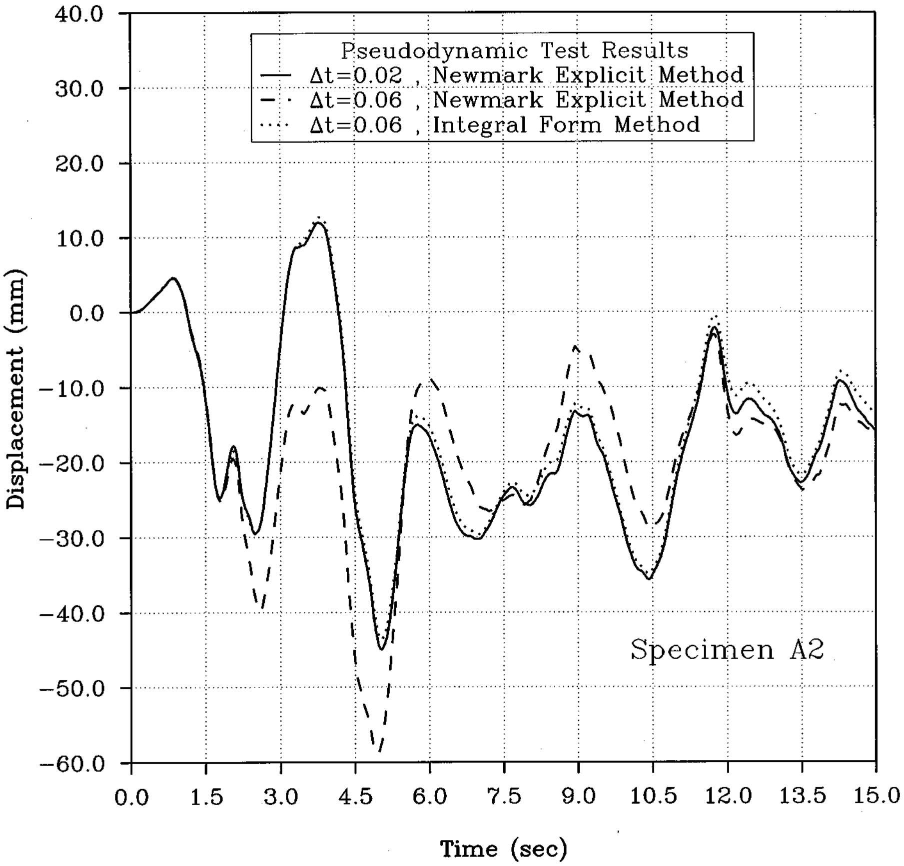

Figure 13. Pseudodynamic response to 0·15 g El Centro Record

Test results for non-linear systems

Specimens A1 and A2 were tested non-linearly. The measured stiffness of both specimens was found to be 24·45 kN/mm and the masses were prescribed as 0·518 and 2·072 kg in order to achieve the systems having the natural period of 1 s for Specimen A1 and 2 s for Specimen A2, respectively.

Test results for the system with 1 s natural period are shown in Figure 12 and the results for the 2 s natural period system are plotted in Figure 13. For the purposes of comparison, the pseudodynamic test results obtained by using the Newmark explicit method with a time step of 0·02 s are considered as the "exact" responses. In Figure 12, two test results were also plotted for the Newmark explicit method and its integral form using a time step of 0·04 s. This figure clearly indicates that the integral form of Newmark explicit method gives very reliable test results while it significantly deviate from the reliable results if the Newmark explicit method is employed. This phenomenon can also be found in Figure 13 where a larger time step of 0·06 s is adopted for pseudodynamic tests using both integration methods. Very significant errors are found in both figures if the Newmark explicit method is applied for both two test systems. The very satisfactory test results obtained from the integral form of Newmark explicit method can be explained by two considerations. First, the time integration reduces the high-frequency content of the accelerogram. Secondly, the potential linearization errors are effectively eliminated by the time integration of the restoring forces.

CONCLUSIONS

The proposed time-integration technique for the Newmark explicit method has been successfully implemented. It is confirmed that the integral form of Newmark explicit method exhibits a much less error propagation characteristics than the original form does. The superior performance of the integral scheme in capturing the rapid changes of dynamic loading is illustrated by the pseudodynamic tests of TADAS devices. These experimental tests also confirm that the adverse linearization errors occurred in the Newmark explicit method can be effectively eliminated by the proposed integral approach. Thus, it is highly recommended that the integral scheme be considered in replacing the original Newmark explicit method for pseudodynamic testings. In addition, the proposed time-integration procedure appears very suitable for pseudodynamic testings using any other available direct integration methods. Currently, the integral scheme for a mixed explicit–implicit integration procedure is being implemented for a series of substructure pseudodynamic tests.

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